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CERTAIN CONDITION ON THE SECOND FUNDAMENTAL FORM OF CR SUBMANIFOLDS OF MAXIMAL CR DIMENSION OF COMPLEX PROJECTIVE SPACE

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ABSTRACT

We study m-dimensional real submanifolds M with $(m-1)$-dimensional maximal holomorphic tangent subspace in complex projective space. On these manifolds there exists an almost contact structure F which is naturally induced from the ambient space and in this paper we study the condition $h(FX,Y)-h(X,FY)=g(FX,Y)\eta$, $\eta\in T^{\perp}(M)$, on the almost contact structure F and on the second fundamental form h of these submanifolds and we characterize certain model spaces in complex projective space.

1. Introduction

The purpose of the present paper is to study CR submanifolds of maximal CR dimension with certain condition on the naturally induced almost contact structure and on the second fundamental form of these submanifolds. Namely, let

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M^m be a real submanifold of the complex manifold $(\overline{M}^{m+p}, \overline{g})$ with complex structure J. If, for any $x \in M$, the tangent space $T_x(M)$ of M at x satisfies $\dim_{\mathbb{R}}(JT_x(M) \cap T_x(M)) = m - 1$, then M is called a CR submanifold of maximal CR dimension. It follows that there exists a unit vector field ξ normal to M such that $JT_x(M) \subset T_x(M) \oplus span\{\xi_x\}$, for any $x \in M$. A real hypersurface is a typical example of a CR submanifold of maximal CR dimension and the generalization of some results which are valid for real hypersurfaces to CR submanifolds of maximal CR dimension may be expected. In the real hypersurface case and in particular when $\overline{M}$ is a Kähler manifold, many results have been obtained. See, for example, [16] for the fundamental definitions and results and for further references. On the other hand, for arbitrary codimension, less detailed results are known but may be expected. For example, we refer to [7], [6], [8] and [9].

Let M be a real hypersurface of an almost Hermitian manifold $\overline{M}$. In [21] Y. Tashiro showed that in this case M is equipped with an almost contact metric structure F naturally induced by the almost Hermitian structure on $\overline{M}$. This has been a fertile field for many authors, in particular when $\overline{M}$ is a complex space form. See [3], [11] and [18] for more details and further references. Above all, M. Okumura, S. Montiel and A. Romero gave a geometric meaning of the commutativity of the second fundamental tensor A of the real hypersurface of a complex space form and its induced almost contact structure F ([15], [18]). Above all, M. Kon ([13]) proved that a connected complete real hypersurface in a complex projective space which satisfies the condition $FA + AF = kF$ for some constant $k \neq 0$ is congruent to some model spaces of type B. Namely, although the complex projective space and complex hyperbolic space can be regarded as the simplest after the spaces of constant curvature, they impose significant restrictions on the geometry of their hypersurfaces. For instance, they do not admit umbilic hypersurfaces (a fact first noted in [23]) and their geodesic spheres do not have constant curvature. H. B. Lawson ([14]) was the first to exploit the idea of regarding the complex projective space $\mathbb{C}P^{\frac{m+1}{2}}$ as a projection from the sphere S^{m+2} with fiber S^1 to study a hypersurface in $\mathbb{C}P^{\frac{m+1}{2}}$ by lifting it to an S^1-invariant hypersurface of the sphere. Further, Takagi's classification given in [20] of the homogeneous real hypersurfaces of complex projective space was important in its own right, but it also identified a whole list of hypersurfaces, gave them names (type $A1$, type B, etc.), and focused attention on them. Other geometers began to study them and to derive new characterizations of various subsets of the list. For example, much of the work has involved finding sufficient conditions for a hypersurface to be one of the "standard examples," characterized by the fact that they have two or three distinct constant principle curvatures. We recall here that the description of the only complete real hypersurfaces in complex projective space with three distinct constant principal curvatures as tubes is given in [3]. In [10] we continued this study concerning CR submanifolds of maximal CR dimension in complex Euclidean space and we obtained a complete classification of those which satisfy certain condition on the almost contact structure naturally induced from the ambient space and on the second fundamental form. In this paper our purpose is to study the case when the ambient manifold is a complex projective space and to prove

Main Theorem: *Let M be a complete m-dimensional CR submanifold of maximal CR dimension of a complex projective space $\mathbb{C}P^{\frac{m+p}{2}}$. If the condition*

$$h(FX,Y) - h(X,FY) = g(FX,Y)\eta, \quad \eta \in T^{\perp}(M)$$

is satisfied, where F and h are the induced almost contact structure and the second fundamental form of M, respectively, then F is a contact structure and M is congruent to a geodesic sphere or to a tube over the complex quadric, or there exists a geodesic hypersphere of $\mathbb{C}P^{\frac{m+p}{2}}$ such that M is its invariant submanifold.

2. CR submanifolds of maximal CR dimension of a Kähler manifold

Let $\overline{M}$ be an $(m+p)$-dimensional Kähler manifold with Kähler structure $(J,\bar{g})$ and let M be an m-dimensional real submanifold of $\overline{M}$ with the immersion $\imath$ of M into $\overline{M}$, whose metric g is induced from $\bar{g}$ in such a way that $g(X,Y) = \overline{g}(\imath X, \imath Y)$, where $X, Y \in T(M)$.

Next, it is known that, for any $x \in M$, the subspace $H_x(M) = JT_x(M) \cap T_x(M)$, is the maximal J-invariant subspace of the tangent space $T_x(M)$ at x. It is called the holomorphic tangent space to M at x. In general, the dimension of $H_x(M)$ varies with x (see [8], for example), but if the subspace $H_x(M)$ has constant dimension for any $x \in M$, the submanifold M is called the Cauchy-Riemann submanifold or briefly CR submanifold and the constant complex dimension of $H_x(M)$ is called the CR dimension of M ([17], [24]). It is well-known that a real hypersurface is one of the typical examples of CR submanifolds whose CR dimension is $\frac{m-1}{2}$, where m is the dimension of a hypersurface. It is easily seen that if M is a CR submanifold in the sense of Bejancu's definition given in [1], M is also a CR submanifold in the sense of the above-given definition. In the case when M is a CR submanifold of CR dimension $\frac{m-1}{2}$, these definitions coincide. On the other hand, when the CR dimension is less than $\frac{m-1}{2}$, the converse is wrong. We refer to [8] for more details and examples of CR submanifolds of maximal CR dimension.

In the sequel we consider CR submanifolds of maximal CR dimension. Consequently M is odd-dimensional and there exists a unit vector field ξ_x normal to $T_x(M)$ such that $JT_x(M) \subset T_x(M) \oplus \operatorname{span}\{\xi_x\}$, for any $x \in M$. Defining a skew–symmetric $(1,1)$-tensor F from the tangential projection of J by

$$J\imath X = \imath FX + u(X)\xi, \tag{1}$$

for any $X \in T(M)$, the Hermitian property of $\bar{g}$ and J implies that the subbundle $T_1^\perp(M) = \{\eta \in T^\perp(M) | \overline{g}(\eta, \xi) = 0\}$ is J-invariant, from which it follows

$$J\xi = -\imath U, \quad g(U, X) = u(X), \quad U \in T(M). \tag{2}$$

Therefore, we denote the orthonormal basis of $T^\perp(M)$ by $\xi, \xi_1, \dots, \xi_q$, $\xi_{1^*}, \dots, \xi_{q^*}$, from now on, where $\xi_{a^*} = J\xi_a$ and $q = (p-1)/2$.

Further, applying J to (1), (2) and comparing the tangential and the normal part to M, we obtain

$$F^2X = -X + u(X)U, \tag{3}$$

$$u(FX) = 0\,, \quad FU = 0. \tag{4}$$

We continue by recalling some general preliminary facts concerning submanifolds. Let us denote by $\overline{\nabla}$ and ∇ the Riemannian connection of $\overline{M}$ and M, respectively. They are related by the following well-known Gauss formula

$$\overline{\nabla}_{\imath X} \imath Y = \imath \nabla_X Y + h(X, Y)\,, \tag{5}$$

where h denotes the second fundamental form

$$h(X, Y) = g(AX, Y)\xi + \sum_{a=1}^{q} \{g(A_a X, Y)\xi_a + g(A_{a^*} X, Y)\xi_{a^*}\}, \tag{6}$$

and A, A_a, A_{a^*}, $a = 1, \dots, q$, are the shape operators corresponding to the normals ξ, ξ_a, ξ_{a^*}, respectively.

Since the ambient manifold is a Kähler manifold, differentiating covariantly relation (1), using (2), (5), and Weingarten formula $\overline{\nabla}_{\imath X}\xi = -\imath AX + D_X\xi = -\imath AX + \sum_{a=1}^{q}\{s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*}\}$, for ξ, where D is the normal connection induced from $\overline{\nabla}$ in the normal bundle of M, and comparing the tangential and the normal part, we get

$$(7)\qquad \begin{gathered}(\nabla_Y F)X = u(X)AY - g(AY,X)U\,, \quad (\nabla_Y u)(X) = g(FAY,X)\,,\\ \nabla_X U = FAX\,.\end{gathered}$$

As stated before, the submanifold M is odd-dimensional and dim $M = m = 2l+1$. If on M there exists a function ρ which takes a value zero nowhere, satisfying

$$(8)\qquad du(X,Y) = \rho g(FX,Y),$$

for any tangent vector fields X, Y, that is, if for the Kähler form ω of $\overline{M}$ we have $du(X,Y) = \rho\omega(\imath X, \imath Y) = \rho(\omega\circ\imath)(X,Y)$, then, since F has rank $2l$, we easily obtain $u\wedge(du)^l \neq 0$. This shows that u is a contact form of M and M is a contact manifold. In this sense, we call the submanifold M, whose induced almost contact structure (F,u,U,g) satisfies (8), a contact submanifold. From now on we suppose that the dimension of the contact submanifold M is greater than 3. Then, from (7) and (8), we easily see that the almost contact structure (F,u,U,g) is contact if and only if there exists a function ρ which takes a value zero nowhere and satisfies a relation:

$$(9)\qquad FA + AF = \rho F\,.$$

3. CR submanifolds of complex projective space satisfying certain condition

In this section we study CR submanifolds M^m of maximal CR dimension of a Kähler manifold, especially of a complex projective space $\mathbb{C}P^{\frac{m+p}{2}}$, which satisfy the condition

$$(10)\qquad h(FX,Y) - h(X,FY) = g(FX,Y)\eta, \quad \eta\in T^\perp(M)$$

for all $X, Y \in T(M)$. Since F is skew-symmetric endomorphism acting on $T(M)$, using relation (6) and setting $\eta = \rho\xi + \sum_{a=1}^{q}(\rho^a\xi_a + \rho^{a^*}\xi_{a^*})$, $q = (p-1)/2$, it follows that relation (10) is equivalent to

$$AFX + FAX = \rho FX, \tag{11}$$

$$A_a FX + FA_a X = \rho^a FX, \quad A_{a^*} FX + FA_{a^*} X = \rho^{a^*} FX, \quad a = 1, \dots, q. \tag{12}$$

Using relation (9) it follows that CR submanifolds of maximal CR dimension of Kähler manifolds which satisfy the condition (10) are contact submanifolds.

Let us begin with several important consequences of the condition (10).

Lemma 1 ([10]): *Let M be an m-dimensional CR submanifold of maximal CR dimension of a Kähler manifold $\overline{M}$. If the condition (11) is satisfied, then U is an eigenvector of the shape operator A with respect to distinguished normal vector field ξ, at any point of M.*

Lemma 2 ([10]): *Let M be an m-dimensional CR submanifold of maximal CR dimension of a Kähler manifold $\overline{M}$. If the condition (10) is satisfied, it follows $\rho^a = 0$, $\rho^{a^*} = 0$, namely*

$$FA_a + A_a F = 0, \quad FA_{a^*} + A_{a^*} F = 0, \quad a = 1, \dots, q. \tag{13}$$

Remark 1: In the rest of the paper we will assume that $\rho \neq 0$, since the case $\rho = 0$ reduces the condition (10) to $h(FX, Y) - h(X, FY) = 0$, which we have considered in [5]. Moreover, in [7], Lemma 3.1., the authors proved that $\rho \neq 0$ is constant.

When the ambient manifold is a Kähler manifold, differentiating covariantly relation $J\xi_a = \xi_{a^*}$ and using (1), (2) and Weingarten formulas, we obtain

$$s_{a^*}(X) = g(A_a X, U), \qquad s_a(X) = -g(A_{a^*} X, U), \tag{14}$$

$$A_{a^*} X = FA_a X - s_a(X)U, \qquad A_a X = -FA_{a^*} X + s_{a^*}(X)U, \tag{15}$$

for $X \in T(M)$ and $a = 1, \dots, q$. Further, using (13), (3), (4) and (14), we obtain

$$A_a U = s_{a^*}(U)U, \qquad A_{a^*} U = -s_a(U)U, \tag{16}$$

$$s_a(X) = s_a(U)u(X), \qquad s_{a^*}(X) = s_{a^*}(U)u(X). \tag{17}$$

If, moreover, $\overline{M}$ is a complex space form, the Codazzi equations for normal vectors ξ_a, become

$$(18)\quad \begin{aligned}(\nabla_X A_a)Y - (\nabla_Y A_a)X &= s_a(Y)AX - s_a(X)AY + \sum_{b=1}^{q}\{s_{ab}(X)A_bY - s_{ab}(Y)A_bX\}\\ &\quad + \sum_{b=1}^{q}\{s_{ab^*}(X)A_{b^*}Y - s_{ab^*}(Y)A_{b^*}X\}, \quad a = 1,\dots,q-1,\end{aligned}$$

where s_{ab}, s_{ab^*} are the coefficients of the normal connection D. We also obtain the equation similar to (18), for normal vectors ξ_{a^*}, $a^* = 1,\dots,q-1$ and where the corresponding coefficients of the normal connection s_{a^*b}, $s_{a^*b^*}$ satisfy

$$(19)\quad s_{a^*b} = -s_{ab^*}, \quad s_{a^*b^*} = s_{ab}.$$

Also, using Ricci-Khüne formula, Gauss equation, (1) and (2), we obtain

$$(20)\quad \begin{aligned}\bar{g}(\overline{R}(\imath X, \imath Y)\xi_a, \xi) &= g(AA_aX, Y) - g(A_aAX, Y) + (\nabla_X s_a)(Y)\\ &\quad - (\nabla_Y s_a)(X) + \sum_{b=1}^{q}[s_b(Y)s_{ba}(X)\\ &\quad + s_{b^*}(Y)s_{b^*a}(X) - s_b(X)s_{ba}(Y) - s_{b^*}(X)s_{b^*a}(Y)] = 0.\end{aligned}$$

Using Gauss and Weingarten formulas, a routine, but long, calculation, yields the following extremely useful result.

Proposition 1: *Let M be a complete m-dimensional CR submanifold of CR dimension $\frac{m-1}{2}$ of a complex space form. If the condition* (10) *is satisfied, then the distinguished normal vector field ξ is parallel with respect to the normal connection.*

Proof. Let us compute $g((\nabla_X A_{a^*})Y - (\nabla_Y A_{a^*})X, U)$ in the following two ways. First, differentiating the first equation of (15) and using (7) and (15), we obtain

$$(21)\quad \begin{aligned}g((\nabla_X A_{a^*})Y, U) &= (\nabla_X F)g(A_aY, U) + g(F(\nabla_X A_a)Y, U) - (\nabla_X s_a)(Y)\\ &= -g(A_aAX, Y) + \alpha s_{a^*}(U)u(X)u(Y) - (\nabla_X s_a)(Y),\end{aligned}$$

where $AU = \alpha U$, after Lemma 1. Reversing X and Y and subtracting thus yields

$$(22)\quad g((\nabla_X A_{a^*})Y - (\nabla_Y A_{a^*})X, U) = g((AA_a - A_aA)X, Y) - (\nabla_X s_a)(Y) + (\nabla_Y s_a)(X).$$

Substituting (18) into (22) and using (20), we obtain

$$(23)\qquad g((AA_a - A_aA)X, Y) = 0, \quad \text{for all} \quad X, Y \in T(M).$$

Next differentiating the second equation of (16) and using (7) and (11), we obtain

$$(24)\quad g((\nabla_X A_{a^*})Y - (\nabla_Y A_{a^*})X, U) + g(A_{a^*}FAX, Y) - g(A_{a^*}FAY, X) = -X(s_a(U))u(Y) + Y(s_a(U))u(X) + s_a(U)\rho g(FX, Y),$$

Further using (11) and (17), relation (24) reads

$$\sum_{b=1}^{q}\{s_{a^*b}(X)s_{b^*}(Y) - s_{a^*b}(Y)s_{b^*}(X) - s_{a^*b^*}(X)s_b(Y) + s_{ab^*}(Y)s_b(X)\} - g((A_aA - A_aA)X, Y)$$
$$(25)\qquad = -X(s_a(U))u(Y) + Y(s_a(U))u(X) - \rho s_a(U)g(FX, Y).$$

Moreover, by Lemma 1 and relations (4) and (13), it follows $g(A_aFAX, U) - g(A_aFAU, X) = 0$. Replacing Y by U in relation (24) and using (17), we obtain

$$(26)\quad X(s_a(U)) = U(s_a(U))u(X)$$
$$-\sum_{b=1}^{q}\{s_{a^*b}(X)s_{b^*}(U) - s_{a^*b^*}(X)s_b(U) - u(X)(s_{a^*b}(U)s_{b^*}(U) + s_{a^*b^*}(U)s_b(U))\}.$$

Combining relation (26) with (25) yields

$$(27)\qquad g((AA_a - A_aA)X, Y) = \rho s_a(U)g(FX, Y).$$

Thus (23) and (27) imply $s_a(U) = 0$ and consequently, from (17) we conclude $s_a(X) = 0$. In entirely the same way, we obtain $s_{a^*} = 0$, which completes the proof. ∎

In the remainder of this section we assume the ambient manifold $\overline{M}$ to be the complex projective space.

As a real hypersurface of a Kähler manifold is a typical example of CR submanifolds of maximal CR dimension. First, we recall some known results on real hypersurfaces of complex projective space. For a real hypersurface, the condition (10) reduces to (11) and in this case the shape operator A of the hypersurface has at most three distinct eigenvalues. Since there exist neither totally geodesic real hypersurfaces nor totally umbilical real hypersurfaces of complex projective space ([23]), under the condition (11), the shape operator A has two or three distinct eigenvalues.

If A has only two distinct eigenvalues, α and λ, then A has the form $AX = \lambda X + (\alpha - \lambda)u(X)U$ and $FAX + AFX = 2\lambda FX$. In this case the real hypersurface M^m of $\mathbb{C}P^{\frac{m+1}{2}}$ is isometric with a geodesic hypersphere $M_0(m, r)$ which is defined by

$$\pi\left\{(z_0, \ldots, z_l) \in \mathbb{C}^{l+1} : |z_0| = \cos^2 r, \sum_{j=1}^{l} |z_j|^2 = \sin^2 r\right\}$$

where $l = \frac{m+1}{2}$ and π is the Hopf fibration $S^{m+2} \to \mathbb{C}P^{\frac{m+1}{2}}$.

If A has three distinct eigenvalues, α, λ_1 and λ_2, then the multiplicities of α, λ_1 and λ_2 are 1, $l-1$ and $l-1$, respectively. The distributions defined by $D_1 = \{X \in T(M)\colon AX = \lambda_1 X\}$, $D_2 = \{X \in T(M)\colon AX = \lambda_2 X\}$ satisfy $FD_1 = D_2$, $FD_2 = D_1$ and such type of real hypersurfaces of complex projective space are called **real hypersurfaces of type B** (see [20]). Consequently, M is isometric with the model space $M(m, t)$ which is defined by

$$\pi\left\{(z_0, \ldots, z_l) \in \mathbb{C}^{l+1} : \left|\sum_{j=0}^{l} z_j^2\right|^2 = t, \sum_{j=0}^{l} |z_j|^2 = 1\right\},$$

where t is a fixed positive number $0 < t < 1$. Cecil and Ryan in [3] proved that $M(m, t)$ is a tube around the complex quadric.

We continue the study in this section by considering the case when M is a complete m-dimensional CR submanifold of maximal CR dimension of a complex projective space $\mathbb{C}P^{\frac{m+p}{2}}$ which satisfies the condition (10). Using Proposition 1 , the first equation of (15) becomes $A_{a^*} = FA_a$ and, for any tangent vector Y, we obtain

$$A_a FAY - AFA_a Y = 0.$$

Therefore, using (11), (13) and (23), we conclude

$$\rho A_a FY - 2A_a AFY = 0. \tag{28}$$

For another eigenvector X, orthogonal to U, with the corresponding eigenvalue λ, since X can be written as $X = FY$and $AFY = \lambda FY = \lambda X$, we can rewrite

(28) as

$$(\rho - 2\lambda)A_a X = 0. \tag{29}$$

Consequently, relation (29) implies that the proof of the Main Theorem falls naturally into two parts.

Let us first consider the case when $\rho \neq 2\lambda$. It follows from (29) that $A_a X = 0$, for all $X \perp U$. Combining (14) and Proposition 1 gives

$$g(A_a U, Y) = s_{a^*}(Y) = 0, \quad \text{for all } Y \in T(M)$$

and therefore $A_a U = 0$. Hence, taking into account that $A_a X = 0$, it follows $A_a = 0$, $a = 1, \dots, q$.

In the same manner we can see that $A_{a^*} = 0$, $a = 1, \dots, q$. We have thus proved:

Lemma 3: *Let M be a complete m-dimensional CR submanifold of maximal CR dimension of a complex projective space $\mathbb{C}P^{\frac{m+p}{2}}$. If the the condition* (10) *is satisfied and $\rho \neq 2\lambda$, where X is another eigenvector of A, orthogonal to U, with the corresponding eigenvalue λ, then $A_a = 0 = A_{a^*}$, $a = 1, \dots, q$, $q = \frac{p-1}{2}$, where A, A_a, A_{a^*} are the shape operators for the normals ξ, ξ_a, ξ_{a^*}, respectively.*

Making use of this result, we prove

Theorem 1: *Let M be a complete m-dimensional CR submanifold of maximal CR dimension of a complex projective space $\mathbb{C}P^{\frac{m+p}{2}}$. If the condition* (10) *is satisfied and $\rho \neq 2\lambda$, where X is another eigenvector of A, orthogonal to U, with the corresponding eigenvalue λ, then there exists a totally geodesic complex projective subspace $\mathbb{C}P^{\frac{m+1}{2}}$ of $\mathbb{C}P^{\frac{m+p}{2}}$ such that M is a real hypersurface of $\mathbb{C}P^{\frac{m+1}{2}}$.*

Proof. First, let us define $N_0(x) = \{\xi \in T_x^{\perp}(M)\colon A_\xi = 0\}$ and let $H_0(x)$ be the maximal J-invariant subspace of $N_0(x)$, that is, $H_0(x) = JN_0(x) \cap N_0(x)$. Then, using Lemma 3, it follows that $N_0(x) = \operatorname{span}\{\xi_1(x), \dots, \xi_q(x), \xi_{1^*}(x), \dots, \xi_{q^*}(x)\}$. Moreover, by the second equation of (7), we obtain $JN_0(x) = N_0(x)$ and consequently $H_0(x) = \operatorname{span}\{\xi_1(x), \dots, \xi_q(x), \xi_{1^*}(x), \dots, \xi_{q^*}(x)\}$. Hence the orthogonal complement $H_1(x)$ of $H_0(x)$ in $T^{\perp}(M)$ is spanned by ξ. Further, it follows from Lemma 1 that ξ is parallel with respect to the normal connection, and we can apply the codimension reduction theorem for real submanifolds of complex projective space ([19]) and conclude that there exists real $(m+1)$-dimensional totally geodesic complex projective subspace of $\mathbb{C}P^{\frac{m+p}{2}}$, such that M is its real hypersurface. ∎

Consequently, since using Theorem 1, the submanifold M can be regarded as a real hypersurface of $\mathbb{C}P^{\frac{m+1}{2}}$ and we can apply the results of a real hypersurface theory, especially [13].

In what follows we denote by $\imath_1$ the immersion of M into $\mathbb{C}P^{\frac{m+1}{2}}$, and by $\imath_2$ the totally geodesic immersion of $\mathbb{C}P^{\frac{m+1}{2}}$ into $\mathbb{C}P^{\frac{m+p}{2}}$. Then, from the Gauss formula (5), it follows that $\nabla'_{\imath_1 X}\imath_1 Y = \imath_1\nabla_X Y + g(A'X,Y)\xi'$, where ξ' is a unit normal vector field to M in $\mathbb{C}P^{\frac{m+1}{2}}$ and A' is the corresponding shape operator. Thus, by using the Gauss equation and $\imath = \imath_2\cdot\imath_1$, we derive

$$(30)\quad \overline{\nabla}_{\imath_2\cdot\imath_1 X}\imath_2\cdot\imath_1 Y = \imath_2\nabla'_{\imath_1 X}\imath_1 Y + \bar{h}(\imath_1 X,\imath_1 Y) = \imath_2(\imath_1\nabla_X Y + g(A'X,Y)\xi'),$$

since $\mathbb{C}P^{\frac{m+1}{2}}$ is totally geodesic in $\mathbb{C}P^{\frac{m+p}{2}}$. Further, comparing relation (30) with relation (5), it follows that $\xi = \imath_2\xi'$ and $A = A'$. As $\mathbb{C}P^{\frac{m+1}{2}}$ is a complex submanifold of $\mathbb{C}P^{\frac{m+p}{2}}$, with the induced complex structure J', we have $J\imath_2 X' = \imath_2 J'X'$, $X' \in T(\mathbb{C}P^{\frac{m+1}{2}})$. Thus, from (1) it follows

$$(31)\qquad J\imath X = \imath_2 J'\imath_1 X = \imath F'X + \nu'(X)\imath_2\xi' = \imath F'X + \nu'(X)\xi$$

and therefore, we conclude that $F = F'$ and $\nu' = u$. Since Kon (see Theorem 3.3. [13]) proved that if for a real hypersurface in complex projective space with almost contact structure F' and second fundamental form A' the condition $F'A' + A'F' = kF'$ is fulfilled for some constant $k \neq 0$, then it is congruent to $M_0(m,r)$ or $M(m,t)$. Combining this with Theorem 1 completes the proof of

Theorem 2: *Let M be a complete m-dimensional CR submanifold of maximal CR dimension of a complex projective space $\mathbb{C}P^{\frac{m+p}{2}}$. If the condition* (10) *is satisfied and $\rho \neq 2\lambda$, where X is another eigenvector of A, orthogonal to U, with the corresponding eigenvalue λ and F and h are the induced contact structure and the second fundamental form of M, respectively, then M is congruent to $M_0(m,r)$ or $M(m,t)$.*

Finally, let us consider the case when $\rho = 2\lambda$. Then, if X is an eigenvector of A, orthogonal to U, with the corresponding eigenvalue λ, using (11), it follows $AFX = (\rho-\lambda)FX$. Namely, FX is an eigenvector of A with the corresponding eigenvalue $\rho-\lambda = 2\lambda-\lambda = \lambda$. Therefore, the only eigenvalues of A are α and λ and we can use the following

THEOREM 3 ([7]): *Let M be an $m(> 2p-1, p \geq 2)$-dimensional real submanifold of a complex projective space $\mathbb{C}P^{\frac{m+p}{2}}$ with maximal holomorphic tangent subspace of dimension $m-1$. If M is a contact submanifold in the sense of* (8) *and the normal field ξ is parallel with respect to the normal connection, and if the shape operator A corresponding to ξ has at most two eigenvalues, then there exists a geodesic hypersphere S' of $\mathbb{C}P^{\frac{m+p}{2}}$ such that M is an invariant submanifold of S'.*

Moreover, on this occasion we do not need the restriction on the dimension, since in paper [7] it was necessary only for proving that the multiplicity of the eigenvalue α corresponding to U is one. However, for $\rho = 2\lambda$, the multiplicity of α is always one. Namely, let us suppose that $Y \perp U$ is another eigenvector corresponding to α. Then relation (11) implies that FY is an eigenvector of A corresponding to $\rho - \alpha$. As A has only two distinct eigenvalues, $\rho - \alpha = \lambda$ and $\rho - \alpha = \alpha$, both cases imply that A has only one eigenvalue, which is impossible, since Lemma 3.6. [7] states that if A has only one eigenvalue α, it follows that $\overline{M}$ is a complex Euclidean space.

Summarizing this with Theorem 2 completes the proof of the Main Theorem.

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